# CH1 BASIC ELEMENTARY FIELDS

→ Fundamental constituents are described by relativistic quantum mechanics laws: Quantum Field Theory

These field can be classified in terms of their transformation properties; each field corresponds to an irrep of the Lorentz group.

## ⊙ A real scalar field $\phi(x)$:

→ Spin 0: $\phi(x) \mapsto \phi'(x') \overset{!}{=} \phi(\Lambda^{-1}x')$ with $x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$

→ It has 1 dof

→ The possible Lorentz invariant bilinears are: $\partial_\mu \phi \partial^\mu \phi$ and $\phi^2$

→ The free scalar Lagrangian is $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m \phi^2$

↳ The EOM is $(\Box + m^2)\phi = 0$, the Klein-Gordon equation. Its solution is $\phi(x) = \int d^3k \{ a(k)\, e^{ikx} + a^\dagger(k)\, e^{-ikx} \}$ ⇝ $\phi^\dagger = \phi$

$\phi$ ⟶ • ⟶ $\phi$

## ⊙ A complex scalar field:

→ Spin 0: $\phi(x) \mapsto \phi'(x') \overset{!}{=} \phi(\Lambda^{-1}x')$, with $\begin{cases} \phi(x) = \mathrm{Re}\{\phi\} + \mathrm{Im}\{\phi\} \\ \phi^\dagger = \mathrm{Re}\{\phi\} - \mathrm{Im}\{\phi\} \end{cases}$

→ It has 2 dof

→ The possible Lorentz invariant bilinears are: $\partial_\mu \phi^\dagger \partial^\mu \phi$, $\phi^\dagger \phi$.

They're hermitian to conserve probability

→ The free field Lagrangian is: $\mathcal{L} = \partial^\mu \phi^\dagger \partial_\mu \phi - \underbrace{m^2 \phi^\dagger \phi}$

↳ The EOM is $(\Box + m^2)\phi = 0$ & $(\Box + m^2)\phi^\dagger = 0$ degenerate mass

↳ Solution:

$$\phi(x) = \int d^3k \{ a(k)\, e^{-ikx} + b^\dagger(k)\, e^{ikx} \}$$

$$\phi^\dagger(x) = \int d^3k \{ a^\dagger(k)\, e^{ikx} + b(k)\, e^{-ikx} \} \rightsquigarrow \phi^\dagger \neq \phi$$

## ⊙ A vector field $A^\mu(x)$ :

→ Spin 1 : transforms as $A'^\mu(x') = \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x')$

→ The Lorentz invariant bilinears are $\partial_\mu A^\nu \partial^\mu A_\nu$, $\partial_\mu A^\nu \partial_\nu A^\mu$, $A_\mu A^\mu$.
↳ Only $F_{\mu\nu}F^{\mu\nu}$ allows for a positive defined Hamiltonian, with
$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$$

→ The free field Lagrangian is: $\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu$
↳ EOM: $\partial_\mu F_{\mu\nu} + m^2 A_\nu = 0$
→ For $m \neq 0$, EOM $\Rightarrow (\Box + m^2) A_\nu = 0$
→ For $m = 0$, we can place ourself in the Lorentz gauge $\partial_\mu A^\mu = 0$ to recover the K-G EOM.
↳ Solution:
$$A_\mu(x) = \int d^3k \sum_\lambda \left\{ a^\lambda(k)\, \varepsilon^\lambda_\mu(k)\, e^{-ikx} + a^{\lambda\dagger}(k)\, \varepsilon^{\lambda *}_\mu(k)\, e^{ikx} \right\}$$

Since $\partial_\mu A^\mu = 0$, $\varepsilon^\lambda_\mu k^\mu = 0 \Rightarrow \lambda \in \{1,2,3\}$
→ We have 2 transverse polarization mode
$\varepsilon^{1,2}_\mu = (0, \vec{e}_{1,2}(k))$ such that $\vec{e}_{1,2}(k) \cdot \vec{k} = 0$
and 1 longitudinal polarization mode
$\varepsilon^3_\mu = \left( \frac{|\vec{k}|}{m}, \frac{E}{m} \cdot \frac{\vec{k}}{|\vec{k}|} \right)$, unphysical for $m=0$

→ $\varepsilon^3_\mu$ doesn't couple to $J^\mu$ throught $A_\mu J^\mu \in \mathcal{L}_I$. Indeed,
$\partial_\mu J^\mu = 0 \Rightarrow k_\mu J^\mu = 0 \Leftrightarrow E J^0 = \vec{k} \cdot \vec{J}$ so that
$$\varepsilon^3_\mu J^\mu = \frac{|k|}{m} J^0 - \frac{E}{m} \frac{k \cdot \bar{J}}{|k|} = k \cdot \bar{J} \left( \frac{|k|}{E.m} - \frac{E}{m} \cdot \frac{1}{|k|} \right)$$
$$= J^0 \left( \frac{|k|}{m} - \frac{E^2}{m|k|} \right) = \frac{J^0}{|k|} \left( \frac{k^2 - E^2}{m} \right) = \frac{J^0 m}{|k|} \xrightarrow{m \to 0} 0$$

## ③ Dirac spinor $\psi(x)$:

→ Spin 1/2 : $\psi(x) \mapsto \psi'(x') = e^{-\frac{i}{2}\omega_{\alpha\beta}S^{\alpha\beta}} \cdot \psi(\Lambda^{-1}x')$
where $S^{\alpha\beta} = \frac{i}{4}[\gamma^\alpha, \gamma^\beta]$ are the Lorentz algebra generators.

↳ In chiral components; in the Weyl representation:
$\psi = \psi_L + \psi_R = P_L\psi + P_R\psi = \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix}$

$$\Rightarrow \chi_{L,R}(x) \mapsto \chi'_{L,R}(x') = \exp\Big\{ \underbrace{\mp \tfrac{1}{2}\omega_{0i}\sigma^i}_{3\text{ boosts}} \underbrace{- \tfrac{i}{4}\omega_{ij}\epsilon^{ij}{}_k\sigma^k}_{3\text{ rotations}} \Big\} \chi_{L,R}(\Lambda^{-1}x')$$

→ The Lorentz invariant bilinears are $i\bar\psi\not{\partial}\psi$ and $\bar\psi\psi$.

→ The Dirac Lagrangian is $\mathcal{L} = \bar\psi(i\not{\partial} - m)\psi$

↳ EoM: $(i\not{\partial} - m)\psi = 0$ the Dirac equation. In Weyl rep.,

$$\begin{cases} i(\partial_t - \sigma\cdot\nabla)\chi_L = m\chi_R \\ i(\partial_t + \sigma\cdot\nabla)\chi_R = m\chi_L \end{cases}$$ the Weyl equations

$\chi_L \xrightarrow{\; m \;} \chi_R$ ($\bar\psi_R\psi_L$) $\qquad \chi_R \xrightarrow{\; m \;} \chi_L$ ($\bar\psi_L\psi_R$)

↳ Solution:

$$\psi(x) = \int d^3k \sum_{s=1}^{2} \left\{ a_k^s u^s(k) e^{-ikx} + b_k^{s\dagger} v^s(k) e^{ikx} \right\}$$

$$\bar\psi(x) = \int d^3k \sum_{s=1}^{2} \left\{ a_k^{s\dagger} \bar u^s(k) e^{ikx} + b_k^s \bar v^s(k) e^{-ikx} \right\}$$

with $u^s(k) = \begin{pmatrix} \sqrt{k\cdot\sigma}\,\xi^s \\ \sqrt{k\cdot\bar\sigma}\,\xi^s \end{pmatrix}$; $v^s(k) = \begin{pmatrix} \sqrt{k\cdot\sigma}\,\xi^s \\ -\sqrt{k\cdot\bar\sigma}\,\xi^s \end{pmatrix}$ and $\xi^\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ spin up, $\xi^\downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ spin down

and $\sigma^\mu \equiv (\mathbb{1}, \sigma^i)$ ; $\bar\sigma^\mu = (\mathbb{1}, -\sigma^i)$

Spin ↑, ↓ are physical states (⇐ conservation of spin). These are superposition of $\psi_L$ and $\psi_R$, which are only physical if $m = 0$. Hence, helicity $\propto \vec{S}\cdot\vec{p}$ is ≠ from chirality if $m \neq 0$.

→ Weyl rep : $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}$

→ Clifford algebra : $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$

→ $\underset{\text{rotation}}{S^{ij}} = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$ and $\underset{\text{boost}}{S^{0i}} = \frac{i}{2}\begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}$

→ Since $\eta'_{\mu\nu} = \Lambda^\alpha{}_\mu\Lambda^\beta{}_\nu\eta_{\alpha\beta} = \eta_{\mu\nu}$, $\Lambda^\mu{}_\nu\gamma^\nu = \Lambda^{-1}_{1/2}\gamma^\mu\Lambda_{1/2}$

## ⊙ Massless Weyl spinor:

→ If $m=0$, $\chi_L$ and $\chi_R$ do not mix. One could have just 2 dof:
$\chi_L$ or $\chi_L$ but not both.
↳ $\chi_L$ transforms as L-field ≡ particle
↳ $\chi_L^c \equiv -i\sigma^2 \chi_L^*$ as a R-field ≡ anti-particle

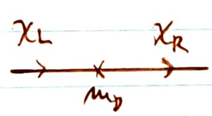
→ The Lorentz invariant, $\mathcal{L}$ are the same as for Dirac. The solutions too, reducing to

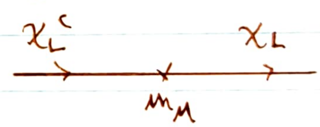
$$u_\downarrow(\vec{k}) = \sqrt{2\omega_k}\begin{pmatrix} \xi_\downarrow \\ 0 \end{pmatrix} : \text{anti-// to } \vec{k} \rightsquigarrow \begin{pmatrix} \chi_L \\ 0 \end{pmatrix}$$
$$v_\uparrow(\vec{k}) = -\sqrt{2\omega_k}\begin{pmatrix} 0 \\ \xi_\downarrow \end{pmatrix} : \text{// to } \vec{k} \rightsquigarrow \begin{pmatrix} 0 \\ \chi_L^c \end{pmatrix}$$

## ⊙ Massive Majorana Spinor:

→ A single $\psi_L$ or $\psi_R$ can have a mass: $\chi_L^{c\dagger}\chi_L$ is Lorentz inv.

↳ $\mathcal{L}^{\text{Maj-Mass}} = -\frac{1}{2}\left(\chi_L^{c\dagger}\chi_L + \chi_L^\dagger \chi_L^c\right) m_M$ is Lorentz inv and hermitian.

→ 2 types of mass: 




→ The physical state $\psi_M = \begin{pmatrix} \chi_L \\ \chi_L^c \end{pmatrix} = P_L\psi_M + P_R\psi_M = \psi_M^c$
where $\psi_M^c \equiv C\,\bar{\psi}_M^T$ with $C \equiv \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix} = i\gamma^2\gamma^0$

→ $\mathcal{L} = \bar{\psi}_L i\partial\!\!\!/ \psi_L - \frac{m_M}{2}\left(\bar{\psi}_L^c \psi + \bar{\psi}_L \psi_L^c\right)$
$\quad = \frac{1}{2}\bar{\psi}_M i\partial\!\!\!/ \psi_M - \frac{m_M}{2}\bar{\psi}_M\psi_M$

→ EOM: $\psi_M = \int d^3\vec{k} \sum_s \left\{ a_k^s u^s(k) e^{-ikx} + a_p^{s\dagger} v^s(p) e^{ikx} \right\}$